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On the Smoothness of Best Spline Approximations

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1. INTRODUCTION

This paper is a sequel to [4] which was concerned with the problem of approximating a prescribed function $f \in C[a,b]$ in the uniform norm by Tchebycheffian Spline Functions (TSF's) with free knots. For convenience, we repeat the definition of this class of splines. Let $\{w_i(t)\}_0^n$ be $n+1$ positive functions on $[a,b]$ with $w_i \in C^{n-i}$, $i = 0,1,\ldots,n$, and let $\{u_i(t)\}_0^n$ be the associated Extended Complete Tchebycheff (ECT) system generated by the weights $\{w_i\}$, i.e.,

$$u_i(t) = w_0(t) \int_a^t w_1(\xi_1) \int_a^{\xi_1} w_2(\xi_2) \ldots \int_a^{\xi_{i-1}} w_i(\xi_i)\, d\xi_i \ldots d\xi_1, \tag{1.1}$$

(cf. [2, 3, 4, 5]). Denote by π_n the class of u-polynomials $\sum_{i=0}^n a_i u_i(t)$. We are interested in approximating continuous functions by functions of the class

$\mathscr{S}_{n,k} = \{s(t) |$ there exist $a = x_0 < x_1 < \ldots < x_{r+1} = b$ and integers $m_1, \ldots, m_r$ with $1 \leqslant m_i \leqslant n+1$ and $\sum_{i=1}^r m_i = k$, such that $s(t) \in \pi_n$ in each of the intervals (x_i, x_{i+1}) while $s \in C^{n-m_i}$ in an open neighborhood of x_i, $1 \leqslant i \leqslant r\}$ (1.2)

of TSF's of degree n with some k knots (counting multiplicities) in $[a,b]$. Here we recall that a spline s of degree n is said to have a knot of multiplicity m at the point x if $s \in C^{n-m}$ in an open neighborhood of x but s is in no higher continuity class there.

The class $\mathscr{S}_{n,k}$ consists precisely of the functions

$$s(t) = \sum_{i=0}^n b_i u_i(t) + \sum_{i=1}^r \sum_{j=1}^{m_i} c_{ij} \phi_{n-j+1}(t; x_i), \qquad \sum_{i=1}^r m_i = k, \tag{1.3}$$

where for $0 \leqslant l \leqslant n$,

$$\phi_l(t;x) = \begin{cases} w_0(t) \int_x^t w_1(\xi_1) \ldots \int_x^{\xi_{l-1}} w_l(\xi_l)\, d\xi_l \ldots d\xi_1, & t \geqslant x \\ 0 & t < x. \end{cases} \tag{1.4}$$

When $w_i(t) \equiv i$, $i = 1,2,\ldots,n$, and $w_0(t) \equiv 1$, then $\{u_i(t)\}_0^n$ become $\{t^i\}_0^n$, $\phi_l(t;x) = (t-x)_+^l$, and the class $\mathscr{S}_{n,k}$ reduces to the set of all polynomial splines of degree n with some k knots, counting multiplicity.

In [4], it was shown that $\mathcal{S}_{n,k}$ is a reasonable class of splines to consider for the purpose of uniform approximation. In particular, it was shown that for every prescribed $f \in C[a,b]$ there exists a best approximation $s^* \in \mathcal{S}_{n,k}$ of f in the uniform norm:

$$\|s^* - f\|_\infty = \max_{a \leq t \leq b} |s^*(t) - f(t)| = \min_{s \in \mathcal{S}_{n,k}} \|s - f\|_\infty.$$

As examples quoted in [4] and [5] show, a prescribed $f \in C[a,b]$ need not have a unique best approximation, and since the class $\mathcal{S}_{n,k}$ allows for splines with multiplicity $n+1$, f may even possess discontinuous best approximations in $\mathcal{S}_{n,k}$. In order to facilitate the discussion of uniqueness and characterization properties of best approximations, the following stronger existence theorem was also obtained in [4].

THEOREM 1.1. *Let $f \in C[a,b]$ and $n \geqslant 1$. Then there exists a best uniform approximation of f in $\mathcal{S}_{n,k}$ which is also in $C[a,b]$.*

The purpose of this paper is to investigate further the smoothness properties of best approximating splines in $\mathcal{S}_{n,k}$. Specifically, we shall show that if $n \geqslant 2$ and $f \in C^1[a,b]$, then f possesses a best approximation in $\mathcal{S}_{n,k}$ (k an arbitrary nonnegative integer) which is also of class $C^1[a,b]$. On the other hand, if n, k, p are integers with $p \geqslant z$, $k > n - p \geqslant 0$, there exists a function $f \in C^\infty[a,b]$ which possess no best approximation in $\mathcal{S}_{n,k}$ of continuity class $C^p[a,b]$ (see Theorem 3.8). This negative result is somewhat unexpected, in view of the positive results in the preservation of continuity and differentiability of f.

2. EXISTENCE OF BEST APPROXIMATIONS IN $\mathcal{S}_{n,k}$ WHICH ARE $C^1[a,b]$

This section is devoted to the following analog of Theorem 1.1.

THEOREM 2.1. *Let $n \geqslant 2$ and $k \geqslant 0$ be integers, and suppose $f \in C^1[a,b]$. Then there exists a best approximation of f in $\mathcal{S}_{n,k}$ which is also in $C^1[a,b]$.*

Proof. Suppose $s \in \mathcal{S}_{n,k}$ is a best approximation of f. One exists by Theorem 1.1, and, moreover, we may even assume it is continuous; i.e., it has no knots of multiplicity $n+1$. Now, if s also exhibits no knots of multiplicity n, then it is *a priori* of class $C^1[a,b]$ and there is nothing to prove. Thus, we consider henceforth only the case where s possesses n-tuple knots at some points in $[a,b]$. Restricting our attention to just one such point $z \in (a,b)$, we may assume that $s(t)$ has the representation (cf. (1.3))

$$s(t) = p(t) + \sum_{i=1}^{n} a_i \phi_i(t; z), \qquad p \in \pi_n, \qquad a_1 \neq 0 \tag{2.1}$$

for t in a small neighborhood of z. By a trivial change of variables we may also assume $z = 0$.

To establish the existence of a best approximation of f in $\mathscr{S}_{n,k}$ which is in $C^1[a,b]$, we shall replace s locally by a spline

$$\tilde{s}(t) = p(t) + A\phi_n(t;x_1) + \sum_{i=3}^{n} A_i\,\phi_i(t;0) + B\phi_n(t;x_2) \tag{2.2}$$

with $x_1 < 0 < x_2$ and $x_2 - x_1$ arbitrarily small. We intend to accomplish this in such a way that $s(t) \equiv \tilde{s}(t)$ for $t \notin [x_1, x_2]$ and so that $\tilde{s}(t)$ also provides a best approximation of f in $\mathscr{S}_{n,k}$. The first requirement leads to the equation

$$s(t) - \tilde{s}(t) = \sum_{i=1}^{n} a_i\,\phi_i(t;0) - A\phi_n(t;x_1) - \sum_{i=3}^{n} A_i\,\phi_i(t;0) - B\phi_n(t;x_2) \equiv 0, \tag{2.3}$$

for $t \geqslant x_2$. We now need the following:

LEMMA 2.2. *For* $t \geqslant \max(0,x)$,

$$\phi_r(t;x) = \phi_r(t;0) + \sum_{i=0}^{r-1} \alpha_i(r;x)\,\phi_i(t;0),$$

where

$$\alpha_i(r;x) = -\int_0^x w_{i+1}(\xi_{i+1}) \int_x^{\xi_{i+1}} w_{i+2}(\xi_{i+2}) \cdots \int_x^{\xi_{r-1}} w_r(\xi_r)\,d\xi_r \ldots d\xi_{i+1},$$
$$i = 0, 1, \ldots, r-1.$$

Proof. For $t \geqslant \max(0,x)$,

$$\phi_r(t;x) = w_0(t) \int_0^t w_1(\xi_1) \int_x^{\xi_1} w_2(\xi_2) \cdots \int_x^{\xi_{r-1}} w_r(\xi_r)\,d\xi_r \ldots d\xi_1$$
$$- w_0(t)\left[\int_0^x w_1(\xi_1) \int_x^{\xi_1} w_2(\xi_2) \cdots \int_x^{\xi_{r-1}} w_r(\xi_r)\;d\xi_r \ldots d\xi_1\right].$$

Similarly,

$$w_0(t) \int_0^t w_1(\xi_1) \int_x^{\xi_1} w_2(\xi_2) \cdots \int_x^{\xi_{r-1}} w_r(\xi_r)\,d\xi_r \ldots d\xi_1$$
$$= w_0(t) \int_0^t w_1(\xi_1) \int_0^{\xi_1} w_2(\xi_2) \cdots \int_x^{\xi_{r-1}} w_r(\xi_r)\,d\xi_r \ldots d\xi_1$$
$$- w_0(t) \int_0^t w_1(\xi_1)\,d\xi_1 \left[\int_0^x w_2(\xi_2) \cdots \int_x^{\xi_{r-1}} w_r(\xi_r)\,d\xi_r \ldots d\xi_2\right].$$

Repeating this process clearly leads to the desired expansion, and the proof of Lemma 2.2 is complete.

Substituting from Lemma 2.2 in (2.3), we obtain the equation ($\alpha_n \equiv 1$)

$$\sum_{i=1}^{n} a_i \phi_i(t;0) - A \sum_{i=0}^{n} \alpha_i(n;x_1)\phi_i(t;0) - \sum_{i=3}^{n} A_i \phi_i(t;0)$$

$$- B \sum_{i=0}^{n} \alpha_i(n;x_2)\phi_i(t;0) \equiv 0, \qquad \text{for } t \geqslant x_2. \tag{2.4}$$

Since the $\{\phi_i(t;0)\}_0^n$ are known to be linearly independent (see e.g., [2]), (2.4) is equivalent to the equations

$$\left.\begin{aligned} A\alpha_0 + B\beta_0 &= 0 \\ A\alpha_1 + B\beta_1 &= a_1 \\ A\alpha_2 + B\beta_2 &= a_2 \\ A_i + A\alpha_i + B\beta_i &= a_i \qquad i = 3,4,\ldots,n, \end{aligned}\right\} \tag{2.5}$$

where for convenience we have written $\alpha_i = \alpha_i(n;x_1)$ and $\beta_i = \alpha_i(n;x_2)$, for $i = 0,1,\ldots,n$. We claim this system of equations can be satisfied with $x_1 < 0 < x_2$ and $x_2 - x_1$ arbitrarily small. To see this, we first notice that for $x_1 < 0 < x_2$, α_i and β_i satisfy the easily verifiable properties

$$\begin{aligned} \alpha_i &> 0. \qquad i = 0,1,\ldots,n, \\ (-1)^{n-i}\beta_i &> 0, \end{aligned} \tag{2.6}$$

This assures that $\alpha_0\beta_1 - \alpha_1\beta_0 \neq 0$ for all choices of $x_1 < 0 < x_2$. Thus, the first two equations of (2.5) can be solved for any $x_1 < 0 < x_2$ and yield

$$A = \frac{a_1\beta_0}{\alpha_1\beta_0 - \alpha_0\beta_1}, \qquad B = \frac{a_1\alpha_0}{\alpha_0\beta_1 - \beta_0\alpha_1}. \tag{2.7}$$

Substituting in the third equation of (2.5), we see that it will be satisfied if and only if

$$I(x_1,x_2) \stackrel{d}{=} \frac{\beta_0\alpha_2 - \alpha_0\beta_2}{\alpha_1\beta_0 - \alpha_0\beta_1} = \frac{a_2}{a_1}.$$

Straightforward application of L'Hospital's rule shows that α_2/α_0, α_1/α_0 and α_2/α_1 all approach $+\infty$ as $x_1 \uparrow 0$, while $\beta_2/\beta_0 \to +\infty$, and β_1/β_0, $\beta_2/\beta_1 \to -\infty$ as $x_2 \downarrow 0$. Hence, for arbitrary $\epsilon > 0$, if $x_2 = \epsilon$, then

$$I(x_1,x_2) = \frac{\beta_0 \dfrac{\alpha_2}{\alpha_0} - \beta_2}{\beta_0 \dfrac{\alpha_1}{\alpha_0} - \beta_1} \to \frac{\alpha_2}{\alpha_1} \to \infty \qquad \text{as } x_1 \uparrow 0.$$

Similarly, if $x_1 = -\epsilon$, then $I(x_1,x_2) \to -\infty$ as $x_2 \downarrow 0$. We conclude that for arbitrarily large $K > 0$ there exist $-\epsilon \leqslant \hat{x}_1 < 0 < \hat{x}_2 \leqslant \epsilon$ such that $I(\hat{x}_1, \epsilon) = K$,

$I(-\epsilon, \hat{x}_2) = -K$. Since $I(x_1, x_2)$ is continuous in x_1 and x_2, this implies there exists a point (x_1, x_2) on the line joining $(-\epsilon, \hat{x}_2)$ and $(\hat{x}_1, \epsilon)$, with the property $I(x_1, x_2) = a_2/a_1$. For this choice of $-\epsilon \leqslant x_1 < 0 < x_2 \leqslant \epsilon$, the first three equations of (2.5) are satisfied. The remaining equations of (2.5) are then solved trivially by choosing $A_i = a_i - A\alpha_i - B\beta_i$, $i = 3, 4, \ldots, n$.

We have now succeeded in constructing $\tilde{s}(t)$ which agrees identically with $s(t)$ for $t \notin [x_1, x_2]$. Clearly, $\tilde{s} \in \mathscr{S}_{n,k}$ since the n-tuple knot at 0 of s has been split into two simple (order 1) knots at x_1 and x_2 and an n-2-tuple knot at 0. Moreover, $\tilde{s}(t)$ is now of class C^2 near 0 by construction. To complete the proof of Theorem 2.1, it remains only to verify that $\tilde{s}$ is still a best approximation of f in $\mathscr{S}_{n,k}$.

We consider only the case where a_1 in (2.1) is positive (the case $a_1 < 0$ is analogous). First we notice that

$$s(0) - f(0) < B_{n,k} = \min_{s \in \mathscr{S}_{n,k}} \|f - s\|_\infty. \tag{2.8}$$

Indeed, suppose this is not the case; i.e., $s(0) - f(0) = B_{n,k}$. Then $(s-f)'(0-) = (p-f)'(0) \geqslant 0$ and for small $t \geqslant 0$,

$$\begin{aligned}[s(t) - f(t)] &= [s(0) - f(0)] + [s'(0) - f'(0)]\,t + 0(t^2) \\ &= B_{n,k} + [a_1 \cdot w_0(0)\, w_1(0) + p'(0) - f'(0)]\,t + 0(t^2) \\ &> B_{n,k}.\end{aligned}$$

(Note: $\phi_i'(t;0)|_0 = 0$ for $i > 1$ while $\phi_1'(t;0)|_0 = w_0(0)\,w_1(0)$.) This contradiction implies (2.8). Now, by (2.6) and (2.7),

$$A = \frac{a_1 \beta_0}{\alpha_1 \beta_0 - \alpha_0 \beta_1} > 0.$$

Since $\tilde{s}(t) - s(t) = A\phi_n(t; x_1)$ for $x_1 \leqslant t \leqslant 0$, it follows that $\tilde{s}(t) > s(t)$ for $x_1 < t \leqslant 0$. Moreover, since $\tilde{s}(t) - s(t)$ is continuous and

$$\tilde{s}(t) - s(t) = -Bw_0(t) \int_{x_2}^{t} w_1(\xi_1) \ldots \int_{x_2}^{\xi_{n-1}} w_n(\xi_n)\, d\xi_n \ldots d\xi_1 \qquad \text{for } 0 < t \leqslant x_2,$$

we conclude that $\tilde{s}(t) - s(t) > 0$ for $x_1 < t < x_2$. In addition, for $x_1 \leqslant t \leqslant x_2$,

$$|(\tilde{s} - s)(x)| \leqslant |A|\,|\phi_n(0; x_1)| = \frac{a_1 \beta_0 \phi_n(0; x_1)}{\alpha_1 \beta_0 - \alpha_0 \beta_1} \to 0 \qquad \text{as } x_2 - x_1 \to 0,$$

as can be verified by another simple application of L'Hospital's rule. Combining these facts, we see that by taking $x_2 - x_1$ sufficiently small we can assure that $\tilde{s}(t)$ is also a best approximation of f in $\mathscr{S}_{n,k}$.

3. Negative Results

We begin this section with an example which illustrates that f cannot be expected to possess best approximations in $\mathscr{S}_{n,k}$ for $k > n - p \geqq 0$ which are $C^p[a,b]$, unless $f \in C^p[a,b]$ itself.

EXAMPLE 3.1. Consider approximating $f(t)=\phi_p(t;0)$ by splines of class $\mathscr{S}_{n,k}$, $k>n-p\geqq 0$. The function f is in $\mathscr{S}_{n,n-p+1}\cap C^{p-1}[-1,1]$, but $f\notin C^p[-1,1]$. Clearly, a best approximation of f in $\mathscr{S}_{n,k}$ is f itself, and it is unique. Since $f\notin C^p[-1,1]$, it follows that f cannot possess a best approximation in $\mathscr{S}_{n,k}$ which is $C^p[-1,1]$.

The remainder of our negative results are based on the following example.

EXAMPLE 3.2. For $1\leqslant p\leqslant n$, let $s(t)=\phi_p(t;0)$ on $[-1,1]$, and construct $f\in C^\infty[-1,1]$ such that $(f-s)(t)$ achieves ± 1 with alternating sign at $n+2$ points, in each of the interiors of the intervals

$$I_i=\left[\frac{i}{4n},\frac{i+1}{4n}\right],\qquad i=-4n,-4n+1,\ldots,4n-1.$$

Consider approximating f by splines in $\mathscr{S}_{n,n-p+1}$. Since $f-s$ alternates at least $(n+2)8n-1\geqslant n+2(n-p+1)+1$ times on $[-1,1]$, Theorem 4.2 of [*4*] assures that s is the best approximation of f in $\mathscr{S}_{n,n-p+1}$.

In addition, since $f-s$ alternates $n+2$ times on each of the intervals I_i and s is a u-polynomial there, s restricted to I_i must be the unique best approximation to f by (generalized) polynomials in π_n on I_i.

The usefulness of Example 3.2 is embodied in the fact that we can establish a connection between the existence of smooth best approximations to f and the existence of confined splines; that is splines with bounded support. Indeed, suppose f possesses a best approximation $\tilde{s}\in\mathscr{S}_{n,n-p+1}$ which is also $C^p[-1,1]$. Since $s\notin C^p[-1,1]$, $s\not\equiv\tilde{s}$. Clearly, there must be intervals among the I_i, both to the left and to the right of $[-1/4n,1/4n]$, in which no knot of $\tilde{s}$ appears, no matter how the $n-p+1$ knots of $\tilde{s}$ are distributed in the $8n$ intervals I_i. But on intervals I_i where $\tilde{s}$ exhibits no knots, $\tilde{s}$ must reduce to the unique best approximation on I_i of f in π_n (since $\|f-\tilde{s}\|=\|f-s\|$ by assumption, while s was already seen to be the best approximation in π_n on I_i). Hence we conclude that the spline $\Delta=s-\tilde{s}$ is a confined spline of degree n with at most $2(n-p+1)$ knots.

In [*1*], confined splines with simple knots were discussed. We need the following slight extension of a lemma from [*1*].

LEMMA 3.3. *Let* $s(t)\in\mathscr{C}\mathscr{S}_{n,m}$ *(the class of confined splines of degree n with some m knots, counting multiplicities). Then* $s\not\equiv 0$ *only if* $m\geqslant n+2$.

Proof. Let the knots of $s\in\mathscr{C}\mathscr{S}_{n,m}$ be x_i, with multiplicity m_i, $i=1,2,\ldots,r$, where $\sum_{i=1}^r m_i=m$. Writing multiple knots repeatedly, according to their multiplicity, we may also write these as $y_1 \leqslant y_2 \leqslant \ldots \leqslant y_m$. Since $s(t) \equiv 0$ for $t \geqslant y_m$, there exist $y_m < t_1 < t_2 < \ldots < t_m$ with

$$s(t_l) = \sum_{i=1}^{r} \sum_{j=1}^{m_i} a_{ij} \phi_{n-j+1}(t_l; x_i) = 0, \qquad l = 1, 2, \ldots, m.$$

By Lemma 2.1 of [*4*] (cf. Theorem 1 of [*3*]), if $m \leqslant n+1$, the determinant of this system is positive, and hence $a_{ij} = 0$, $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, m_i$.

An easy consequence of Example 3.2 and Lemma 3.3 is

LEMMA 3.4. *A necessary condition in order that every $f \in C^p$ will possess a best approximation in $\mathcal{S}_{n, n-p+1}$ which is also in C^p, is that $n \geqslant 2p$.*

Proof. Consider the function f described in Example 3.2. It can only possess a C^p best approximation in $\mathcal{S}_{n, n-p+1}$ if there exist confined splines $\Delta \in \mathscr{C}\mathcal{S}_{n, 2(n-p+1)}$. This is possible only if $n + 2 \leqslant 2(n - p + 1)$, which implies $2p \leqslant n$.

This lemma shows that the hypotheses in the positive results of Sections 1 and 2 are necessary.

We need some other intermediate results.

LEMMA 3.5. *Let $s \in \mathscr{C}\mathcal{S}_{n, n+2}$. Then $s \not\equiv 0$ implies $s(t) \neq 0$ for $y_1 < t < y_{n+2}$, where $y_1 \leqslant \ldots \leqslant y_{n+2}$ are the knots of s, repeated according to their multiplicity.*

Proof. Suppose $s(t_1) = 0$ for some $y_1 < t_1 < y_{n+2}$. Choose $y_{n+2} < t_2 < \ldots < t_{n+2}$. Then if $x_1 < \ldots < x_r$ denote the distinct knots of s

$$s(t_l) = \sum_{i=1}^{r} \sum_{j=1}^{m_i} a_{ij} \phi_{n-j+1}(t_l; x_i) = 0, \qquad l = 1, 2, \ldots, n+2.$$

But since $y_i < t_i$, $i = 1, 2, \ldots, n+2$, while $t_1 < y_{n+2}$, Lemma 2.1 of [*4*] assures that the determinant of this system is non-zero and hence $a_{ij} = 0$, $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, m_i$.

LEMMA 3.6. *There exists no confined spline of the form*

$$s(t) = \sum_{i=1}^{r} \sum_{j=1}^{m_i} a_{ij} \phi_{n-j+1}(t; x_i) + \phi_2(t; 0)$$

with $n \geqslant 4$ and $x_1 < 0 < x_r$, $\sum_{i=1}^{r} m_i = n - 1$.

Proof. Suppose such an s exists. Since it is in C^1, Rolle's Theorem assures that

$$s_1(t) = \left[\frac{s(t)}{u_0(t)}\right]' = \sum_{i=1}^{r} \sum_{j=1}^{m_i} a_{ij} \phi_{n-j}^*(t; x_i) + \phi_1^*(t; 0)$$

again vanishes for t outside of $[x_1, x_r]$ and, moreover, $s_1(t_1) = 0$ for some $x_1 < t_1 < x_r$. (The * on the ϕ's indicates that they are defined as in Section 1 but with respect to a different set of weights; here $\{w_i\}_1^n$ instead of $\{w_i\}_0^n$. The reader who is not thoroughly familiar with TSF's and Tchebycheff systems may think of polynomial splines, in which case, *'s are not needed here. If the subscript l of some $\phi_l{}^*$ is negative, then we take $\phi_l{}^* \equiv 0$.) Thus $s_1(t)$ is again a confined TSF. The analysis now divides into two cases.

Case I. Suppose $r = 2$. Then in $(x_1, 0)$ and $(0, x_2)$, s_1 has no knots and can be differentiated as often as desired. Using Rolle's Theorem, $s_2 = (s_1/w_1)'$ and thus also $s_3 = (s_2/w_2)'$ will have a zero in (x_1, x_2), and will vanish for $t \notin [x_1, x_2]$. Since

$$s_3(t) = \sum_{i=1}^{r} \sum_{j=1}^{m_i} a_{ij} \phi_{n-j-2}^*(t; x_i)$$

for all $t \in (x_1, x_2)$, we see that $s_3(t) \in \mathscr{C}\mathscr{S}_{n-3,\,n-1}^*$ and has a zero also in (x_1, x_2). By Lemma 3.5, this is impossible unless $s_3 \equiv 0$.

Case II. Suppose $r \geqslant 3$. Then $m_i \leqslant n - 3$; so s_1 is of class C^2 on $(x_1, 0)$ and $(0, x_2)$. Thus it can be differentiated twice in $(x_1, 0)$ and $(0, x_2)$, and this leads again to an $s_3(t)$ which vanishes identically .

In either case, $s(t)$ reduces to $\phi_2(t; 0)$ which is clearly not a confined spline, and this contradiction proves the lemma.

THEOREM 3.7. *For any $n \geqq 3$ there exists a function $f \in C^\infty[-1, 1]$ such that f has no $C^2[-1, 1]$ best approximations in $\mathscr{S}_{n,\,n-1}$.*

Proof. Consider the function $f(x) \in C^\infty[-1, 1]$ constructed in Example 3.2 (with $p = 2$), whose best approximation in $\mathscr{S}_{n,\,n-1}$ is $s = \phi_2(t; 0)$. Clearly $s \in C^1[-1, 1]$, but by the remarks following Example 3.2, f can possess a C^2 best approximation only if there exists a confined spline of the form

$$\varDelta = \phi_2(t; 0) + g, \qquad g \in \mathscr{S}_{n,\,n-1}.$$

By Lemma 3.3, this requires $n \geqslant 4$, since $\varDelta \in \mathscr{C}\mathscr{S}_{n,\,2n-2}$. On the other hand, for $n \geqslant 4$, Lemma 3.6 shows that there cannot exist confined splines of the form (3.7).

A simple modification of the above method yields the following more complete theorem.

THEOREM 3.8. *Let k, n, p be nonnegative integers.*

(a) If $k \leqslant n-p$, then for any $f \in C[a,b]$ every best approximation of f in $\mathscr{S}_{n,k}$ belongs to $C^p[a,b]$.

(b) If $p \geqslant 2$ and $k > n-p \geqslant 0$, then there exists a function $f \in C^\infty[a,b]$ which possesses no $C^p[a,b]$ best approximation in $\mathscr{S}_{n,k}$.

Proof. Part (a) follows immediately from the definition of $\mathscr{S}_{n,k}$. For (b) consider $s(t) = \phi_p(t;0) + \sum_{i=1}^{l-1} \phi_n(t;i)$, on the interval $[-1,l]$ where $l = k - n + p$. Clearly $s \in \mathscr{S}_{n,k} \cap C^{p-1}[-1,l]$ while $s \notin C^p[-1,l]$. As in Example 3.2 construct $f \in C^\infty[-1,l]$ such that $f - s$ alternates $n+2$ times on the interiors of each of the intervals $I_j = [jh,(j+1)h]$ for $j = -N, -N+1, \ldots, N.l-1$, where $h = 1/N$, $N = k+3$. By the alternation s is a best approximation of f in $\mathscr{S}_{n,k}$. Suppose now that $\tilde{s} \in \mathscr{S}_{n,k}$ belongs to $C^p[-1,l]$ and is also a best approximation of f on $[-1,l]$. No matter how the knots of $\tilde{s}$ are distributed, there exists at least one of the I_j in each of the intervals $[-1+h,-h], \ldots, [l-1+h, l-h]$ with no knot of $\tilde{s}$. Thus (cf. Example 3.2 ff.), $\Delta = s - \tilde{s}$ is identically zero on these I_j, and Δ breaks into l confined splines with supports on disjoint intervals $\Lambda_i \supset [i-h, i+h]$, $i = 0, 1, \ldots, l-1$. We claim $\tilde{s}$ must have at least one knot in each of the Λ_i, $i = 1, 2, \ldots, l-1$. Indeed, if $\tilde{s}$ has no knot in Λ_i, then $s - \tilde{s} \in \mathscr{C}\mathscr{S}_{n,1}$ on Λ_i which implies $s \equiv \tilde{s}$ there in which case it does have a knot at i. We conclude that on Λ_0, Δ is a confined spline of the form $\Delta = \phi_p(t;0) + g$, $g \in \mathscr{S}_{n,n-p+1}$. Applying Rolle's theorem as in Lemma 3.6 it follows that no such confined spline can exist. This contradiction establishes the theorem.

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